SECTION 12

PARTIAL DERIVATIVES OF PRECISION LIGHT TIMES AND QUASAR DELAYS

Contents

12.1 INTRODUCTION

Section 11 gave the formulation for calculating the precision round-trip light time ρ, two versions of the precision one-way light time ρ_1, and the precision quasar delay τ. This section gives the formulation for calculating the partial derivatives of these precision light times and quasar delays with respect to the parameter vector **q**. The parameter vector **q** consists of solve-for parameters and consider parameters. The consider parameters do not affect the parameter solution, but the uncertainties in these parameters contribute to the calculated standard deviations of the solve-for parameters. The partial derivatives of the precision light times ρ and ρ_1 and the precision quasar delay τ with respect to the parameter vector **q** are used in Section 13 to calculate the partial derivatives of the computed values of the observables with respect to **q**.

Section 12.2 gives the high-level equations for calculating the partial derivatives of the position vectors of the participants with respect to the parameter vector **q**. These are Eqs. (8–1) to (8–3) for the position vectors of the participants with each term (a position vector) replaced with the partial derivative of the term with respect to **q**. The partial derivatives of the various terms of the position vectors of the participants with respect to **q** are calculated as specified in the subsections of Section 12.3.

Given the reception time t_3(ET) at the receiver for a spacecraft light-time solution, and the partial derivatives of the position vectors of the participants at their epochs of participation with respect to the parameter vector **q**, Section 12.4 gives equations for the partial derivatives of the transmission time (for a one-way light-time solution) or reflection time t_2(ET) and the transmission time t_1(ET) (for a round-trip light-time solution) with respect to the parameter vector **q**. Similarly, given the reception time t_1(ET) of the quasar wavefront at receiver 1 for a quasar light-time solution, and the partial derivatives of the position vectors of the two receivers at their epochs of participation with respect to **q**, Section 12.4 gives the equation for the partial derivative of the reception time t_2(ET) at receiver 2 with respect to **q**.

The four subsections of Section 12.5 give the formulas for calculating the partial derivatives of the precision round-trip light time ρ, each of the two versions of the precision one-way light time ρ_1, and the precision quasar delay τ with respect to the parameter vector $\mathbf{q}$. Each of these four subsections contains two subsections. The first subsection gives the partial derivative of the precision light time or quasar delay with respect to $\mathbf{q}$ due to the variations of the position vectors of the participants at their epochs of participation with variations in the parameter vector $\mathbf{q}$. These partial derivatives are those derived in Section 12.4. The second subsection gives the partial derivatives of the precision light time or quasar delay with respect to $\mathbf{q}$ due to variations in the observational parameters. These are the parameters that affect the precision light time or quasar delay directly instead of or in addition to changing the position vectors of the participants.

12.2 PARTIAL DERIVATIVES OF POSITION VECTORS OF PARTICIPANTS

For a round-trip spacecraft light-time solution in the Solar-System barycentric space-time frame of reference, the Solar-System barycentric (superscript C) position vectors of the receiver at the reception time t_3, the spacecraft at the reflection time t_2, and the transmitter at the transmission time t_1 are given by Eqs. (8–1) to (8–3). Differentiating these equations with respect to the parameter vector $\mathbf{q}$ gives:

$$\frac{\partial \mathbf{r}_3^{\mathrm{C}}(t_3)}{\partial \mathbf{q}} = \frac{\partial \mathbf{r}_3^{\mathrm{E}}(t_3)}{\partial \mathbf{q}} + \frac{\partial \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_3)}{\partial \mathbf{q}} \tag{12–1}$$

$$\frac{\partial \mathbf{r}_2^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} = \frac{\partial \mathbf{r}_2^{\mathrm{B}}(t_2)}{\partial \mathbf{q}} + \frac{\partial \mathbf{r}_{\mathrm{B}}^{\mathrm{P}}(t_2)}{\partial \mathbf{q}} + \frac{\partial \mathbf{r}_{\mathrm{B,P}}^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} \tag{12–2}$$

$$\frac{\partial \mathbf{r}_1^{\mathrm{C}}(t_1)}{\partial \mathbf{q}} = \frac{\partial \mathbf{r}_1^{\mathrm{E}}(t_1)}{\partial \mathbf{q}} + \frac{\partial \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_1)}{\partial \mathbf{q}} \tag{12–3}$$

For a one-way spacecraft light-time solution, Eq. (12–3) is not used. For a spacecraft light-time solution in the local geocentric space-time frame of reference, Eqs. (12–1) to (12–3) reduce to their first terms and the superscript B in Eq. (12–2) is replaced with E for the Earth.

For a quasar light-time solution in the Solar-System barycentric space-time frame of reference, the partial derivative of the Solar-System barycentric position vector of receiver 1 at the reception time t_1 with respect to the parameter vector $\mathbf{q}$ is given by Eq. (12–3). The partial derivative of the Solar-System barycentric position vector of receiver 2 at the reception time t_2 with respect to $\mathbf{q}$ is given by Eq. (12–3) with each subscript 1 changed to a 2.

The subsections in Section 12.3 describe how the various terms of Eqs. (12–1) to (12–3) are evaluated.

12.3 PARTIAL DERIVATIVES OF SUB-POSITION VECTORS OF PARTICIPANTS

The six subsections of this section describe how the various terms of Eqs. (12–1) to (12–3) are calculated. Each subsection specifies which terms of Eqs. (12–1) to (12–3) it is evaluating and describes how these terms are calculated.

12.3.1 PLANETARY EPHEMERIS PARTIALS

For a spacecraft light-time solution in the Solar-System barycentric space-time frame of reference, evaluation of the partial derivatives (12–1) and (12–3) requires the calculation of the sub-partial derivatives:

$$\frac{\partial \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_3)}{\partial \mathbf{q}}, \quad \frac{\partial \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_1)}{\partial \mathbf{q}} \tag{12–4}$$

Also, if the Solar-System barycentric position vector of the intermediate body B or P at t_2 in Eq. (8–2) is obtained from the planetary ephemeris, the following partial derivative is required to evaluate Eq. (12–2):

$$\frac{\partial \mathbf{r}_{\mathrm{B,P}}^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} \tag{12–5}$$

For a quasar light-time solution, evaluation of Eq. (12–3) for receiver 1 and this same equation with each subscript 1 changed to a 2 for receiver 2 requires the calculation of the partial derivatives:

$$\frac{\partial \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_1)}{\partial \mathbf{q}}, \quad \frac{\partial \mathbf{r}_{\mathrm{E}}^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} \tag{12–6}$$

The partial derivatives (12–4) through (12–6) are calculated as described in Section 3.1.3.

12.3.2 SMALL-BODY EPHEMERIS PARTIALS

For a spacecraft light-time solution in the Solar-System barycentric space-time frame of reference, if the Solar-System barycentric position vector of the intermediate body B in Eq. (8–2) is obtained from the small-body ephemeris instead of from the planetary ephemeris, the last term of Eq. (12–2) (evaluated for small body B, not the center of mass P of a planetary system)

$$\frac{\partial \mathbf{r}_{\mathrm{B}}^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} \tag{12–7}$$

is evaluated by interpolating the small-body partials file for body B at t_2 as described in Sections 3.1.3.1 and 3.1.3.3. In addition to the partial derivatives on this file, calculate the partial derivative with respect to the scaling factor *AU* (the number of kilometers per astronomical unit) from Eq. (3–8) where b = B.

12.3.3 SATELLITE EPHEMERIS PARTIALS

For a spacecraft light-time solution in the Solar-System barycentric space-time frame of reference where the center of integration B for the spacecraft ephemeris or the lander body B is the planet or a satellite of one of the outer planet systems, the second term of Eq. (12–2) is evaluated by interpolating the satellite partials file for the planetary system containing the body B at t_2, as described in Sections 6.5.3 and 3.2.3.

12.3.4 SPACECRAFT EPHEMERIS PARTIALS

For a free spacecraft, the partial derivatives of the space-fixed position vector of the spacecraft (point 2) relative to its center of integration B with respect to the dynamic parameters of the spacecraft ephemeris (term 1 of Eq. 12–2) are obtained by interpolating the PV file for the spacecraft at t_2(ET).

For a spacecraft light-time solution, if the transmitter or the receiver is an Earth satellite instead of a tracking station on Earth, the partial derivative of the geocentric space-fixed position vector of the transmitter or the receiver with respect to the parameter vector $\mathbf{q}$ (term 1 of Eq. 12–3 or 12–1) is obtained by interpolating the geocentric PV file for the transmitter or the receiver.

For a quasar light-time solution, if receiver 1 or receiver 2 is an Earth satellite instead of a tracking station on Earth, the partial derivative of the geocentric space-fixed position vector of receiver 1 or 2 with respect to $\mathbf{q}$ (term 1 of Eq. 12–3 or this same equation with each subscript 1 changed to a 2) is obtained by interpolating the geocentric PV file for receiver 1 or 2.

12.3.5 TRACKING STATION PARTIALS

The partial derivatives of the geocentric space-fixed position vector of a tracking station on Earth with respect to the parameter vector $\mathbf{q}$ are calculated from the formulation given in Section 5.5. For a spacecraft light-time solution, these partial derivatives are calculated at the transmission time t_1 if the transmitter is a tracking station on Earth (term 1 of Eq. 12–3). They are calculated at the reception time t_3 (term 1 of Eq. 12–1) if the receiver is a tracking station on Earth. For a quasar light-time solution, these partial derivatives are calculated at the reception time t_1 at receiver 1 (term 1 of Eq. 12–3) if receiver 1 is a tracking station on Earth. They are calculated at the reception time t_2 at receiver 2 (term 1 of Eq. 12–3 with each subscript 1 changed to a 2) if receiver 2 is a tracking station on Earth.

12.3.6 LANDER PARTIALS

In the spacecraft light-time solution in the Solar-System barycentric space-time frame of reference, if the spacecraft is landed on body B, the partial derivative of the space-fixed position vector of the landed spacecraft (point 2) relative to body B with respect to the parameter vector **q** (term 1 of Eq. 12–2) is calculated from the formulation of Section 6.5.

12.4 TRANSFORMING PARTIAL DERIVATIVES OF POSITION VECTORS OF PARTICIPANTS TO PARTIAL DERIVATIVES OF TRANSMISSION OR RECEPTION TIMES

This section derives partial derivatives of the transmission time or reflection time $t_2(\mathrm{ET})$ and the transmission time $t_1(\mathrm{ET})$ in a spacecraft light-time solution with respect to the parameter vector **q** and the reception time $t_2(\mathrm{ET})$ at receiver 2 in a quasar light-time solution with respect to **q**. These partial derivatives are used in Section 12.5 to calculate partial derivatives of the precision round-trip light time ρ, the precision one-way light time ρ_1, and the quasar delay τ with respect to the parameter vector **q**.

12.4.1 SPACECRAFT LIGHT-TIME SOLUTION

The partial derivatives of the transmission time or reflection time $t_2(\mathrm{ET})$ and the transmission time $t_1(\mathrm{ET})$ with respect to the parameter vector **q** are derived in the Solar-System barycentric space-time frame of reference. Then, the simplifications that apply in the local geocentric space-time frame of reference are noted.

In the Solar-System barycentric space-time frame of reference, the down-leg light-time equation is given by Eq. (8–55). Ignoring the relativistic light-time delay terms gives:

$$t_3(\mathrm{ET}) - t_2(\mathrm{ET}) = \frac{r_{23}}{c} \tag{12–8}$$

From Eqs. (8–57) and (8–58), the down-leg range r_{23} can be calculated from:

$$r_{23}{}^2 = \left[\mathbf{r}_3^{\mathrm{C}}(t_3) - \mathbf{r}_2^{\mathrm{C}}(t_2)\right]^{\mathrm{T}} \left[\mathbf{r}_3^{\mathrm{C}}(t_3) - \mathbf{r}_2^{\mathrm{C}}(t_2)\right] \tag{12–9}$$

where the superscript T indicates the transpose of the vector. In these equations, the reception time $t_3(\mathrm{ET})$ in coordinate time ET is fixed. The Solar-System barycentric (superscript C) position vectors of the receiver (point 3) and the spacecraft (point 2) are functions of the parameter vector $\mathbf{q}$ (see Eqs. 12–1 and 12–2), and the latter vector is also a function of the transmission or reflection time $t_2(\mathrm{ET})$ at the spacecraft. Differentiating Eqs. (12–8) and (12–9) with respect to $\mathbf{q}$ gives:

$$-\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} = \frac{1}{c}\frac{\partial r_{23}}{\partial \mathbf{q}} \tag{12–10}$$

$$\frac{1}{c}\frac{\partial r_{23}}{\partial \mathbf{q}} = \frac{1}{c}\left(\frac{\mathbf{r}_{23}}{r_{23}}\right)^{\mathrm{T}} \left[\frac{\partial \mathbf{r}_3^{\mathrm{C}}(t_3)}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_2^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} - \dot{\mathbf{r}}_2^{\mathrm{C}}(t_2)\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}}\right] \tag{12–11}$$

where $\mathbf{r}_{23}$ is given by Eq. (8–57). Substituting the right-hand side of Eq. (12–10) into Eq. (12–11), solving for $\partial t_2(\mathrm{ET})/\partial \mathbf{q}$, and using the definition (8–60) of $\dot{p}_{23}$ gives:

$$\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} = \frac{\dfrac{1}{c}\left(\dfrac{\mathbf{r}_{23}}{r_{23}}\right)^{\mathrm{T}} \left[\dfrac{\partial \mathbf{r}_2^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} - \dfrac{\partial \mathbf{r}_3^{\mathrm{C}}(t_3)}{\partial \mathbf{q}}\right]}{1 - \dfrac{\dot{p}_{23}}{c}} \tag{12–12}$$

The up-leg light-time equation is also given by Eq. (8–55). Ignoring the relativistic light-time delays gives:

$$t_2(\mathrm{ET}) - t_1(\mathrm{ET}) = \frac{r_{12}}{c} \tag{12–13}$$

From Eqs. (8–57) and (8–58), the up-leg range r_{12} can be calculated from:

$$r_{12}{}^2 = \left[\mathbf{r}_2^{\mathrm{C}}(t_2) - \mathbf{r}_1^{\mathrm{C}}(t_1)\right]^{\mathrm{T}} \left[\mathbf{r}_2^{\mathrm{C}}(t_2) - \mathbf{r}_1^{\mathrm{C}}(t_1)\right] \tag{12–14}$$

In these equations, the Solar-System barycentric position vectors of the spacecraft (point 2) and the transmitter (point 1) are functions of the parameter vector $\mathbf{q}$ (see Eqs. 12–2 and 12–3). Also, the former vector is a function of the reflection time $t_2(\mathrm{ET})$ at the spacecraft, and the latter vector is a function of the transmission time $t_1(\mathrm{ET})$. Differentiating Eqs. (12–13) and (12–14) with respect to $\mathbf{q}$ gives:

$$\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} - \frac{\partial t_1(\mathrm{ET})}{\partial \mathbf{q}} = \frac{1}{c}\frac{\partial r_{12}}{\partial \mathbf{q}} \tag{12–15}$$

$$\frac{1}{c}\frac{\partial r_{12}}{\partial \mathbf{q}} = \frac{1}{c}\left(\frac{\mathbf{r}_{12}}{r_{12}}\right)^{\mathrm{T}} \left[\frac{\partial \mathbf{r}_2^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_1^{\mathrm{C}}(t_1)}{\partial \mathbf{q}} + \dot{\mathbf{r}}_2^{\mathrm{C}}(t_2)\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} - \dot{\mathbf{r}}_1^{\mathrm{C}}(t_1)\frac{\partial t_1(\mathrm{ET})}{\partial \mathbf{q}}\right] \tag{12–16}$$

where $\mathbf{r}_{12}$ is given by Eq. (8–57). Substituting the right-hand side of Eq. (12–15) into Eq. (12–16), replacing $\dot{\mathbf{r}}_2^{\mathrm{C}}(t_2)$ with $\left[\dot{\mathbf{r}}_2^{\mathrm{C}}(t_2) - \dot{\mathbf{r}}_1^{\mathrm{C}}(t_1)\right] + \dot{\mathbf{r}}_1^{\mathrm{C}}(t_1)$, using the definitions (8–59) for $\dot{r}_{12}$ and (8–60) for $\dot{p}_{12}$, and solving for $\partial t_1(\mathrm{ET})/\partial \mathbf{q}$ gives:

$$\frac{\partial t_1(\mathrm{ET})}{\partial \mathbf{q}} = \frac{\dfrac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}}\left(1 - \dfrac{\dot{r}_{12} + \dot{p}_{12}}{c}\right) + \dfrac{1}{c}\left(\dfrac{\mathbf{r}_{12}}{r_{12}}\right)^{\mathrm{T}} \left[\dfrac{\partial \mathbf{r}_1^{\mathrm{C}}(t_1)}{\partial \mathbf{q}} - \dfrac{\partial \mathbf{r}_2^{\mathrm{C}}(t_2)}{\partial \mathbf{q}}\right]}{1 - \dfrac{\dot{p}_{12}}{c}} \tag{12–17}$$

Eqs. (12–12) and (12–17) also apply in the local geocentric space-time frame of reference. However, the up-leg and down-leg unit vectors, $\dot{r}_{12}$, $\dot{p}_{12}$, and $\dot{p}_{23}$ are all calculated from geocentric vectors obtained from the geocentric light-time solution instead of from barycentric vectors obtained from the Solar-System barycentric light-time solution. Also, the partial derivatives of the position vectors of the three participants with respect to $\mathbf{q}$ are the first terms of Eqs. (12–1) to (12–3), which are referred to the Earth (E).

12.4.2 QUASAR LIGHT-TIME SOLUTION

The partial derivative of the reception time $t_2(\mathrm{ET})$ at receiver 2 (a tracking station on Earth or an Earth satellite) with respect to the parameter vector $\mathbf{q}$ is derived in the Solar-System barycentric space-time frame of reference. Note that the quasar light-time solution is only obtained in this frame of reference.

The quasar light-time equation is given by Eqs. (8–91), (8–57), and (8–95). Ignoring the relativistic light-time delay terms gives:

$$t_2(\mathrm{ET}) - t_1(\mathrm{ET}) = \frac{r_{12}}{c} \tag{12–18}$$

From Eqs. (8–57) and (8–95), the range r_{12} is given by:

$$r_{12} = \left[\mathbf{r}_1^{\mathrm{C}}(t_1) - \mathbf{r}_2^{\mathrm{C}}(t_2)\right] \cdot \mathbf{L}_{\mathrm{Q}} \tag{12–19}$$

where the unit vector $\mathbf{L}_{\mathrm{Q}}$ from the Solar System barycenter to the quasar is given by Eqs. (8–92) and (8–93). In Eqs. (12–18) and (12–19), the reception time $t_1(\mathrm{ET})$ in coordinate time ET at receiver 1 is fixed. The Solar-System barycentric (superscript C) position vectors of receiver 1 (point 1) and receiver 2 (point 2) are functions of the parameter vector $\mathbf{q}$ (see Eqs. 12–3 and 12–3 with each subscript 1 changed to a 2), and the latter vector is also a function of the reception time $t_2(\mathrm{ET})$ at receiver 2. Differentiating Eqs. (12–18) and (12–19) with respect to $\mathbf{q}$ gives:

$$\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} = \frac{1}{c}\frac{\partial r_{12}}{\partial \mathbf{q}} \tag{12–20}$$

$$\frac{1}{c}\frac{\partial r_{12}}{\partial \mathbf{q}} = \frac{1}{c}\mathbf{L}_{\mathrm{Q}}{}^{\mathrm{T}}\left[\frac{\partial \mathbf{r}_1^{\mathrm{C}}(t_1)}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_2^{\mathrm{C}}(t_2)}{\partial \mathbf{q}} - \dot{\mathbf{r}}_2^{\mathrm{C}}(t_2)\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}}\right] \tag{12–21}$$

Substituting the right-hand side of Eq. (12–20) into Eq. (12–21), solving for $\partial t_2(\mathrm{ET})/\partial \mathbf{q}$, and using the definition (8–97) for $\dot{p}_{12}$ gives:

$$\frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} = \frac{\frac{1}{c}\mathbf{L}_{\mathrm{Q}}{}^{\mathrm{T}}\left[\frac{\partial \mathbf{r}_1^{\mathrm{C}}(t_1)}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_2^{\mathrm{C}}(t_2)}{\partial \mathbf{q}}\right]}{1 + \frac{\dot{p}_{12}}{c}} \tag{12–22}$$

12.5 PARTIAL DERIVATIVES OF PRECISION LIGHT TIMES AND QUASAR DELAYS WITH RESPECT TO THE PARAMETER VECTOR q

The four subsections of this section give the partial derivatives of the precision round-trip light time ρ, each of the two versions of the precision one-way light light time ρ_1, and the precision quasar delay τ with respect to the parameter vector **q**. Each of the four subsections contains two subsections. The first subsection gives the partial derivative of the precision light time or quasar delay with respect to **q** due to variations in the position vectors of the participants with variations in the parameter vector **q**. These partial derivatives are obtained by differentiating the terms $(r_{23}/c)+(r_{12}/c)$, r_{23}/c, and r_{12}/c of ρ, ρ_1, and τ, respectively. They are calculated using equations derived in Section 12.4. The second subsection gives the observational partial derivatives, which represent direct variations in the precision light time or quasar delay with variations in **q**, holding the position vectors of the participants fixed.

12.5.1 PARTIAL DERIVATIVES OF PRECISION ROUND-TRIP LIGHT TIME ρ

12.5.1.1 Direct Partial Derivatives

The precision round-trip light time ρ is calculated from Eq. (11–7). The position vectors of the participants at their epochs of participation are calculated from Eqs. (8–1) to (8–3). These position vectors are used to calculate the down-leg range r_{23} and the up-leg range r_{12} from Eqs. (12–9) and (12–14). From Eqs. (12–8) and (12–13), the sum of terms one and three of Eq. (11–7), namely $r_{23}/c + r_{12}/c$, is equal to $t_3(\mathrm{ET}) - t_1(\mathrm{ET})$. Hence, the partial derivative of the precision round-trip light time ρ with respect to the parameter vector **q** due to the direct variations in the position vectors of the participants with variations in **q** and due to the indirect variations in the position vectors of the spacecraft at t_2 and the transmitter at t_1 due to variations in t_2 and t_1 with variations in **q** is given by:

$$\frac{\partial \rho}{\partial \mathbf{q}} = -\frac{\partial t_1(\mathrm{ET})}{\partial \mathbf{q}} \tag{12–23}$$

where $\partial t_1(\mathrm{ET})/\partial \mathbf{q}$ is calculated from Eqs. (12–12) and (12–17). In these equations, the partial derivatives of the position vectors of the participants are calculated from Eqs. (12–1) to (12–3).

12.5.1.2 Observational Partial Derivatives

In Eq. (11–7), the down-leg relativistic light-time delay RLT_{23} and the up-leg relativistic light-time delay RLT_{12} contain the factor $(1+\gamma)$. Hence, the partial derivative of the precision round-trip light time ρ with respect to the relativity parameter γ is given approximately by:

$$\frac{\partial \rho}{\partial \gamma} = \frac{RLT_{23} + RLT_{12}}{1+\gamma} \tag{12–24}$$

In the Solar-System barycentric space-time frame of reference, RLT_{23} and RLT_{12} are each calculated as the sum of terms two and three on the right-hand side of Eq. (8–55). In the local geocentric space-time frame of reference, they are calculated from the second term on the right-hand side of Eq. (8–67).

If the receiver is a tracking station on Earth, Eq. (11–7) contains the time difference $(\mathrm{UTC}-\mathrm{ST})_{t_3}$ at the reception time t_3 at the tracking station. If the receiver is an Earth satellite, this time difference is replaced by $(\mathrm{TPX}-\mathrm{ST})_{t_3}$, where TPX is TOPEX master time at the Earth satellite. Similarly, if the transmitter is a tracking station on Earth, Eq. (11–7) contains the time difference $(\mathrm{UTC}-\mathrm{ST})_{t_1}$ at the transmission time t_1 at the tracking station. If the transmitter is an Earth satellite, this time difference is replaced by $(\mathrm{GPS}-\mathrm{ST})_{t_1}$, where GPS is GPS master time at the Earth satellite. Each of these time differences is calculated from the quadratic expression (2–32), as explained after that equation. Changes in either of the time differences at t_3 change $t_3(\mathrm{ET})$ by an equal amount. This change produces a change in t_1 in all of the time scales. Differentiating the sum of Eqs. (12–8) and (12–13) with respect to $t_3(\mathrm{ET})$ gives approximately:

$$\frac{dt_1(\mathrm{ET})}{dt_3(\mathrm{ET})} = 1 - \frac{1}{c}\left(\dot{r}_{12} + \dot{r}_{23}\right) \tag{12–25}$$

where $\dot{r}_{12}$ and $\dot{r}_{23}$ are given by Eq. (8–59). From Eqs. (11–7), (2–32), and (12–25), the partial derivatives of the precision round-trip light time ρ with respect to the *a*, *b*, and *c* quadratic coefficients of the time difference UTC or TPX minus station time ST at the receiver at the reception time t_3 are given by:

$$\frac{\partial \rho}{\partial a} = -\left[1 - \frac{1}{c}\left(\dot{r}_{12} + \dot{r}_{23}\right)\right] \tag{12–26}$$

$$\frac{\partial \rho}{\partial b} = -\left(t_3 - t_0\right)\left[1 - \frac{1}{c}\left(\dot{r}_{12} + \dot{r}_{23}\right)\right] \tag{12–27}$$

$$\frac{\partial \rho}{\partial c} = -\left(t_3 - t_0\right)^2\left[1 - \frac{1}{c}\left(\dot{r}_{12} + \dot{r}_{23}\right)\right] \tag{12–28}$$

where t_3 is the reception time at the receiver in station time ST. From Eqs. (11–7) and (2–32), the partial derivatives of ρ with respect to the *a*, *b*, and *c* quadratic coefficients of the time difference UTC or GPS minus station time ST at the transmitter at the transmission time t_1 are given by:

$$\frac{\partial \rho}{\partial a} = 1 \tag{12–29}$$

$$\frac{\partial \rho}{\partial b} = \left(t_1 - t_0\right) \tag{12–30}$$

$$\frac{\partial \rho}{\partial c} = \left(t_1 - t_0\right)^2 \tag{12–31}$$

where t_1 is the transmission time at the transmitter in station time ST. For three-way data, the *a*, *b*, and *c* coefficients in Eqs. (12–26) to (12–28) are not the same coefficients as those in Eqs. (12–29) to (12–31). For two-way data, the *a*, *b*, and *c* coefficients in Eqs. (12–26) to (12–28) will be the same coefficients as those in Eqs. (12–29) to (12–31) if t_3 and t_1 are in the same time block for *a*, *b*, and *c* for the tracking station or Earth satellite.

From Eq. (11–7), the partial derivative of the precision round-trip light time ρ with respect to the round-trip range bias R_c in meters (for the receiver and time block containing t_3) is given by:

$$\frac{\partial \rho}{\partial R_c} = \frac{1}{10^3 c} \tag{12–32}$$

In Eq. (11–7), the down-leg and up-leg solar corona corrections are calculated from Eq. (10–64) and related equations of Section 10.4. The partial derivatives of the precision round-trip light time ρ with respect to the solve-for *A*, *B*, and *C* coefficients of the solar corona model are given by Eq. (10–76). In this equation, the partial derivatives of the down-leg and up-leg solar corona corrections with respect to the *A*, *B*, and *C* coefficients are given by Eqs. (10–72) to (10–74).

Eq. (11–7) for the precision round-trip light time ρ does not contain tropospheric or charged-particle corrections. These corrections are calculated in the Regres editor and appear in Eq. (10–27) for the media correction $\Delta\rho$ to ρ given by Eq. (11–7). The partial derivatives of ρ with respect to solve-for tropospheric and charged-particle parameters are the partial derivatives of $\Delta\rho$ with respect to these parameters. These partial derivatives are calculated in program Regres, not in the Regres editor.

The partial derivatives of the precision round-trip light time ρ with respect to solve-for constant corrections to the tropospheric zenith dry and wet range corrections $\Delta\rho_{z_{\text{dry}}}$ and $\Delta\rho_{z_{\text{wet}}}$ are calculated from Eq. (10–54) with M changed to T (for troposphere). In this equation, the partial derivatives of the down-leg and up-leg tropospheric range corrections with respect to $\Delta\rho_{z_{\text{dry}}}$ and $\Delta\rho_{z_{\text{wet}}}$ for all tracking stations at a DSN complex or at an isolated tracking station are given by Eq. (10–7). These partial derivatives are the corresponding mapping functions. If the user has selected the Chao mapping functions, they are calculated from Eqs. (10–8) to (10–10). If the user has selected the Niell mapping functions, the dry and wet mapping functions in Eqs. (10–7) are calculated from the formulation specified in Section 10.2.1.3.2.

The ODP user can estimate corrections to the *N* and *D* coefficients of the ionosphere model of Klobuchar (1975). Estimated corrections to *N* and *D* represent corrections to the charged-particle corrections calculated in the Regres editor. The partial derivatives of the precision round-trip light time ρ with respect to corrections to *N* and *D* are calculated from Eq. (10–54) with M changed to I (for ionosphere). In this equation, the partial derivatives of the down-leg and up-leg ionospheric range corrections with respect to the corrections to *N* and *D* that apply for all tracking stations of a DSN complex or for a single isolated tracking station are calculated from Eqs. (10–51) and (10–52) and related equations of Section 10.3.1.

If the receiver or the transmitter is an Earth satellite, the partial derivatives of the down-leg or up-leg tropospheric and ionospheric range corrections with respect to solve-for tropospheric and ionospheric parameters (in Eq. 10–54) should be set to zero.

12.5.2 PARTIAL DERIVATIVES OF PRECISION ONE-WAY LIGHT TIME ρ_1

The precision one-way light time ρ_1 that is used to calculate computed values of one-way doppler (F_1) observables, one-way narrowband spacecraft interferometry (*INS*) observables, and one-way wideband spacecraft interferometry (*IWS*) observables is calculated from Eq. (11–41). It will be seen in the following two subsections that the partial derivatives of this precision one-way light time ρ_1 with respect to the various solve-for parameters are, in general, equal to the down-leg terms of the round-trip partial derivatives given in Section 12.5.1. If detailed descriptions of the calculation of various down-leg terms are omitted in this section, they are the same as given in Section 12.5.1.

12.5.2.1 Direct Partial Derivatives

The partial derivative of the precision one-way light time ρ_1 with respect to the parameter vector **q** due to the direct variations in the position vectors of the participants with variations in **q** and due to the indirect variation in the position vector of the spacecraft at t_2 due to the variation in t_2 with variations in **q** is given by:

$$\frac{\partial \rho_1}{\partial \mathbf{q}} = - \frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} \qquad (12\text{–}33)$$

where $\partial t_2(\mathrm{ET})/\partial \mathbf{q}$ is calculated from Eq. (12–12). In this equation, the partial derivatives of the position vectors of the participants are calculated from Eqs. (12–1) and (12–2).

12.5.2.2 Observational Partial Derivatives

The partial derivative of the precision one-way light time ρ_1 with respect to the relativity parameter γ is given approximately by:

$$\frac{\partial \rho_1}{\partial \gamma} = \frac{RLT_{23}}{1+\gamma} \qquad (12\text{–}34)$$

which is the down-leg term of Eq. (12–24).

The partial derivatives of the precision one-way light time ρ_1 with respect to the a, b, and c quadratic coefficients of the time difference UTC or TPX minus station time ST at the receiver at the reception time t_3 are given by:

$$\frac{\partial \rho_1}{\partial a} = -\left[1 - \frac{\dot{r}_{23}}{c}\right] \qquad (12\text{–}35)$$

$$\frac{\partial \rho_1}{\partial b} = -\left(t_3 - t_0\right)\left[1 - \frac{\dot{r}_{23}}{c}\right] \qquad (12\text{–}36)$$

$$\frac{\partial \rho_1}{\partial c} = -\left(t_3 - t_0\right)^2 \left[1 - \frac{\dot{r}_{23}}{c}\right] \qquad (12–37)$$

These equations are the same as Eqs. (12–26) to (12–28), except that the up-leg range rate $\dot{r}_{12}$ is deleted.

The partial derivatives of the precision one-way light time ρ_1 with respect to the solve-for *A*, *B*, and *C* coefficients of the solar corona model are calculated from Eqs. (10–72) to (10–75).

Eq. (11–41) for the precision one-way light time ρ_1 does not contain tropospheric or charged-particle corrections. These corrections are calculated in the Regres editor and appear in Eq. (10–26) for the media correction $\Delta\rho_1$ to ρ_1 given by Eq. (11–41). The partial derivatives of ρ_1 with respect to solve-for tropospheric and charged-particle parameters are the partial derivatives of $\Delta\rho_1$ with respect to these parameters. These partial derivatives are calculated in program Regres, not in the Regres editor.

The partial derivatives of the precision one-way light time ρ_1 with respect to solve-for constant corrections to the tropospheric zenith dry and wet range corrections $\Delta\rho_{z_{\mathrm{dry}}}$ and $\Delta\rho_{z_{\mathrm{wet}}}$ are calculated as described in Section 12.5.1.2 except that Eq. (10–53) is used instead of Eq. (10–54). The parameters $\Delta\rho_{z_{\mathrm{dry}}}$ and $\Delta\rho_{z_{\mathrm{wet}}}$ are for the isolated receiving station on Earth or the DSN complex that the receiving station is located in. If the receiver is an Earth satellite, set these partial derivatives to zero.

The partial derivatives of the precision one-way light time ρ_1 with respect to corrections to the *N* and *D* coefficients of the ionosphere model of Klobuchar (1975) are calculated as described in Section 12.5.1.2 except that Eq. (10–53) is used instead of Eq. (10–54). The corrections to *N* and *D* are for the isolated receiving station on Earth or the DSN complex that the receiving station is located in. If the receiver is an Earth satellite, set these partial derivatives to zero.

12.5.3 PARTIAL DERIVATIVES OF PRECISION ONE-WAY LIGHT TIME ρ_1 FOR GPS/TOPEX OBSERVABLES

The precision one-way light time ρ_1 (in units of kilometers) that is used to calculate computed values of GPS/TOPEX pseudo-range and carrier-phase observables is calculated from Eq. (11–44).

12.5.3.1 Direct Partial Derivatives

The partial derivative of the precision one-way light time ρ_1 with respect to the parameter vector $\mathbf{q}$ due to the direct variations in the position vectors of the participants with variations in $\mathbf{q}$ and due to the indirect variation in the position vector of the spacecraft at t_2 due to the variation in t_2 with variations in $\mathbf{q}$ is given by Eq. (12–33) multiplied by the speed of light c.

12.5.3.2 Observational Partial Derivatives

The partial derivative of the precision one-way light time ρ_1 with respect to the relativity parameter γ is given approximately by Eq. (12–34) multiplied by the speed of light c.

The partial derivatives of the precision one-way light time ρ_1 with respect to the *a*, *b*, and *c* quadratic coefficients of the time difference GPS (at a GPS receiving station on Earth) or TPX (at the receiving TOPEX satellite) minus station time ST at the receiver at the reception time t_3 are given by Eqs. (12–35) to (12–37) multiplied by the speed of light c.

The partial derivatives of ρ_1 with respect to the *a*, *b*, and *c* quadratic coefficients of the time difference GPS (at the transmitting GPS satellite) minus ST at the transmitter at the transmission time t_2 are given by Eqs. (12–29) to (12–31) with ρ changed to ρ_1, t_1 changed to t_2, and the resulting equations multiplied by the speed of light c.

From Eq. (11–44), the partial derivative of the precision one-way light time ρ_1 with respect to the pseudo-range bias *Bias* or the carrier-phase bias *Bias* (they are separate parameters) in seconds for the receiver and time block containing the reception time t_3 is given by:

$$\frac{\partial \rho_1}{\partial Bias} = c \tag{12–38}$$

The observed values of GPS/TOPEX pseudo-range and carrier-phase observables are calculated as a weighted average of values at two different transmitter frequencies, which eliminates the effects of charged particles. Hence, Eq. (11–44) for ρ_1 does not include solar corona corrections and Eq. (10–26) (multiplied by the speed of light c) for the media correction $\Delta\rho_1$ to ρ_1 given by Eq. (11–44) does not contain charged-particle corrections. Hence, for GPS/TOPEX observables, the partial derivatives of ρ_1 with respect to the A, B, and C coefficients of the solar corona model and the N and D coefficients of the ionosphere model are set to zero.

If the receiver is the TOPEX satellite, the partial derivatives of ρ_1 with respect to solve-for constant corrections to the tropospheric zenith dry and wet range corrections $\Delta\rho_{z_{\mathrm{dry}}}$ and $\Delta\rho_{z_{\mathrm{wet}}}$ are set to zero. If the receiver is a GPS receiving station on Earth, the partial derivatives of ρ_1 with respect to $\Delta\rho_{z_{\mathrm{dry}}}$ and $\Delta\rho_{z_{\mathrm{wet}}}$ for the isolated receiving station on Earth or the DSN complex that the receiving station is located in are calculated as described in Section 12.5.1.2 except that Eq. (10–53) is used instead of Eq. (10–54). Also, the resulting partial derivatives must be multiplied by the speed of light c.

12.5.4 PARTIAL DERIVATIVES OF PRECISION QUASAR DELAY τ

The precision quasar delay τ which is used to calculate the computed values of wideband (*IWQ*) and narrowband (*INQ*) quasar interferometry observables is calculated from Eq. (11–67).

12.5.4.1 Direct Partial Derivatives

The partial derivative of the precision quasar delay τ with respect to the parameter vector $\mathbf{q}$ due to the direct variations in the position vectors of receivers 1 and 2 with variations in $\mathbf{q}$ and due to the indirect variation in the position vector of receiver 2 at t_2 due to the variation in t_2 with variations in $\mathbf{q}$ is given by:

$$\frac{\partial \tau}{\partial \mathbf{q}} = \frac{\partial t_2(\mathrm{ET})}{\partial \mathbf{q}} \tag{12–39}$$

where $\partial t_2(\mathrm{ET})/\partial \mathbf{q}$ is calculated from Eq. (12–22).

12.5.4.2 Observational Partial Derivatives

The first term on the right-hand side of Eq. (11–67), namely r_{12}/c, is calculated from Eqs. (12–18) and (12–19). In the latter equation, the unit vector $\mathbf{L}_\mathrm{Q}$ to the quasar is calculated from Eqs. (8–92) and (8–93). Variations in the right ascension α and declination δ of the quasar in the radio frame (see Eq. 8–92) affect $\mathbf{L}_\mathrm{Q}$. This has a direct effect on the first term on the right-hand side of Eq. (11–67) and an indirect effect due to the change in the reception time t_2 at receiver 2. Accounting for both of these effects, the partial derivatives of the precision quasar delay τ with respect to α and δ of the quasar are given by:

$$\frac{\partial \tau}{\partial \alpha, \delta} = -\frac{\frac{1}{c}\mathbf{r}_{12} \cdot \frac{\partial \mathbf{L}_\mathrm{Q}}{\partial \alpha, \delta}}{1 + \frac{\dot{p}_{12}}{c}} \tag{12–40}$$

where $\mathbf{r}_{12}$ is given by Eq. (8–57) and $\dot{p}_{12}$ is given by Eq. (8–97). From Eq. (8–93),

$$\frac{\partial \mathbf{L}_\mathrm{Q}}{\partial \alpha, \delta} = \left(R_x R_y R_z\right)^\mathrm{T} \frac{\partial \mathbf{L}_{\mathrm{Q}_\mathrm{RF}}}{\partial \alpha, \delta} \tag{12–41}$$

where the frame-tie rotation matrices R_z, R_y, and R_x are given by Eqs. (5–117) to (5–119). From Eq. (8–92),

$$\frac{\partial \mathbf{L}_{\mathrm{QRF}}}{\partial \alpha} = \begin{bmatrix} -\cos\delta \sin\alpha \\ \cos\delta \cos\alpha \\ 0 \end{bmatrix} \tag{12–42}$$

$$\frac{\partial \mathbf{L}_{\mathrm{QRF}}}{\partial \delta} = \begin{bmatrix} -\sin\delta \cos\alpha \\ -\sin\delta \sin\alpha \\ \cos\delta \end{bmatrix} \tag{12–43}$$

In Eq. (8–93), the frame-tie rotation matrices R_z, R_y, and R_x are functions of the frame-tie rotation angles r_z, r_y, and r_x, respectively. The partial derivatives of the precision quasar delay τ with respect to the frame-tie rotation angles r_z, r_y, and r_x can be calculated from Eq. (12–40) with α,δ replaced with r_z, r_y, and r_x. From Eq. (8–93),

$$\frac{\partial \mathbf{L}_{\mathrm{Q}}}{\partial r_z} = \left(R_x R_y \frac{dR_z}{dr_z} \right)^{\mathrm{T}} \mathbf{L}_{\mathrm{QRF}} \tag{12–44}$$

$$\frac{\partial \mathbf{L}_{\mathrm{Q}}}{\partial r_y} = \left(R_x \frac{dR_y}{dr_y} R_z \right)^{\mathrm{T}} \mathbf{L}_{\mathrm{QRF}} \tag{12–45}$$

$$\frac{\partial \mathbf{L}_{\mathrm{Q}}}{\partial r_x} = \left(\frac{dR_x}{dr_x} R_y R_z \right)^{\mathrm{T}} \mathbf{L}_{\mathrm{QRF}} \tag{12–46}$$

where the derivatives of the frame-tie rotation matrices with respect to the frame-tie rotation angles are calculated from Eqs. (5–120) to (5–122).

From Eq. (11–67), the partial derivative of the precision quasar delay τ with respect to the relativity parameter γ is given approximately by:

$$\frac{\partial \tau}{\partial \gamma} = \frac{RLT_{12}}{1+\gamma} \tag{12–47}$$

where the relativistic light-time delay RLT_{12} is calculated as the sum of terms 2 and 3 on the right-hand side of Eq. (8–91).

In Eq. (11–67) for the precision quasar delay τ, the intermediate time UTC at receiver 2 or at receiver 1 is only used if that receiver is a DSN tracking station on Earth. If receiver 2 is an Earth satellite, UTC is replaced with TOPEX master time (TPX) and the constant offset (TAI − TPX) is obtained from the GIN file. Similarly, if receiver 1 is an Earth satellite, UTC is replaced with GPS master time (GPS) and the constant offset (TAI – GPS) is obtained from the GIN file. The time differences UTC or TPX minus station time ST at the reception time t_2 of the quasar wavefront at receiver 2 and the time differences UTC or GPS minus ST at the reception time t_1 of the quasar wavefront at receiver 1 are calculated from the quadratic expression (2–32), as explained after that equation. The change in either of these time differences at t_1 changes $t_1(\mathrm{ET})$ by an equal amount. This change produces a change in t_2 in all of the time scales. Differentiating Eq. (12–18) with respect to $t_1(\mathrm{ET})$ gives approximately:

$$\frac{dt_2(\mathrm{ET})}{dt_1(\mathrm{ET})} = 1 + \frac{\dot{r}_{12}}{c} \tag{12–48}$$

where $\dot{r}_{12}$ is given by Eqs. (8–96) and (8–57). From Eqs. (11–67), (2–32), and (12–48), the partial derivatives of the precision quasar delay τ with respect to the a, b, and c quadratic coefficients of the time difference UTC or GPS minus station time ST at receiver 1 at the reception time t_1 are given by:

$$\frac{\partial \tau}{\partial a} = \left[1 + \frac{\dot{r}_{12}}{c}\right] \tag{12–49}$$

$$\frac{\partial \tau}{\partial b} = \left(t_1 - t_0\right)\left[1 + \frac{\dot{r}_{12}}{c}\right] \tag{12–50}$$

$$\frac{\partial \tau}{\partial c} = \left(t_1 - t_0\right)^2\left[1 + \frac{\dot{r}_{12}}{c}\right] \tag{12–51}$$

where t_1 is the reception time of the quasar wavefront at receiver 1 in station time ST. From Eqs. (11–67) and (2–32), the partial derivatives of τ with respect to the *a*, *b*, and *c* quadratic coefficients of the time difference UTC or TPX minus station time ST at receiver 2 at the reception time t_2 are given by:

$$\frac{\partial \tau}{\partial a} = -1 \tag{12–52}$$

$$\frac{\partial \tau}{\partial b} = -\left(t_2 - t_0\right) \tag{12–53}$$

$$\frac{\partial \tau}{\partial c} = -\left(t_2 - t_0\right)^2 \tag{12–54}$$

where t_2 is the reception time of the quasar wavefront at receiver 2 in station time ST.

From Eq. (11–67), the partial derivatives of the precision quasar delay τ with respect to the *A*, *B*, and *C* coefficients of the solar corona model are calculated from Eq. (10–77). In this equation, the partial derivatives of the down-leg solar corona range corrections for receivers 2 and 1 with respect to the *A*, *B*, and *C* coefficients are given by Eqs. (10–72) to (10–74).

Eq. (11–67) for the precision quasar delay τ does not contain tropospheric or charged-particle corrections. These corrections are calculated in the Regres editor and appear in Eq. (10–30) for the media correction $\Delta\tau$ to τ given by Eq. (11–67). The partial derivatives of τ with respect to solve-for tropospheric and charged-particle parameters are the partial derivatives of $\Delta\tau$ with respect to these parameters. These partial derivatives are calculated in program Regres, not in the Regres editor.

The partial derivatives of the precision quasar delay τ with respect to the solve-for constant corrections to the tropospheric zenith dry and wet range corrections $\Delta\rho_{z_{\text{dry}}}$ and $\Delta\rho_{z_{\text{wet}}}$ and the *N* and *D* coefficients of the ionosphere model of Klobuchar (1975) are calculated from Eq. (10–55). In this equation, the down-leg tropospheric and ionospheric partial derivatives at receivers 2 and 1 are calculated as described in Section 12.5.1.2. If either receiver is an Earth satellite, the tropospheric and ionospheric partials for that receiver should be set to zero.